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LETTER TO THE EDITOR

Burgers' model of turbulence as a stochastic process

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Abstract. A recently proposed mesoscopic description of fluid dynamics leads to a new approach to turbulence. In contrast to the classical statistical theory of turbulence the new approach introduces a probabilistic time evolution of the random velocity governed by a master equation. The mesoscopic approach is explained by means of the $(1+1)$-dimensional Burgers' model of turbulence. By a continuous time stochastic simulation realizations of turbulent velocity fields are generated. Correlation functions and energy spectra are evaluated from appropriate ensemble averages.

It is well known that models of homogeneous turbulence often rely upon statistical tools [1, 2]. In principle, statistical concepts are introduced in the theory only by considering random initial ensembles of velocity fields. However, the time evolution of each member of the ensemble is governed by the deterministic Navier–Stokes equation.

In this letter a mesoscopic approach to homogeneous turbulence is suggested which, in contrast to the classical theory, regards the velocity itself as a discrete stochastic process. The latter is defined by a master equation which replaces the deterministic time evolution expressed by the Navier–Stokes equation. In doing so, an inherently stochastic model of turbulence can be formulated. An initial ensemble of velocity fields evolves probabilistically in time. One of the main advantages of this formulation of fluid dynamics is the following: having defined a stochastic process underlying the dynamics of fluid motion, very efficient methods of stochastic simulation can be employed in order to generate realizations of the random velocity characterizing the turbulent flow.

The new approach is introduced by means of the $(1+1)$-dimensional Burgers' model [3] of homogeneous turbulence which is known to show the main characteristic features of both the nonlinear inertial term and the viscosity term of the Navier–Stokes equation. Burgers proposed the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{R}\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where u is the velocity field and R denotes the Reynolds number, as a simple paradigmatic model of turbulence [4, 5]. The nonlinear inertial term is known to lead to steep gradients of velocity as time evolves. On the other hand, the viscous term tends to smear out sudden changes in the velocity. For large Reynolds numbers the

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two effects lead to the formation of a shock-wave structure which is mainly responsible for the behaviour of the energy spectrum $E(k)$ of Burgers' turbulence model, i.e. $E(k) \sim k^{-2}$.

The mesoscopic approach we suggest in this letter is not based on the solutions of Burgers' equation as a partial differential equation. On the contrary, the model of homogeneous turbulence will be investigated starting from the stochastic process whose ensemble average obeys Burgers' equation. The construction of such a stochastic process has already been given in detail [6–8] and will therefore only be sketched here in order to be self-contained.

In the mesoscopic approach physical space is divided into cells of width δl which are labelled by the integer index λ. The fundamental concept behind this approach is to interpret the velocity field $u(x_\lambda, t)$ appearing in Burgers' equation as the expectation value of a discrete integer valued stochastic process $N_u^\lambda(t)$

$$u(x_\lambda, t) = \delta u \, \langle N_u^\lambda(t) \rangle . \tag{2}$$

In the above equation a mesoscopic velocity scale δu has been introduced which defines the smallest possible changes of the velocity. The stochastic process $N_u^\lambda(t)$ is completely specified by the joint probability distribution $P(\{N_u^\lambda\}, t)$ which is the probability at time t of finding the set of numbers $\{N_u^\lambda\}$. The time evolution of the joint probability distribution P is given by the following master equation [6]

$$\begin{aligned}\frac{\partial P}{\partial t} = {} & \frac{1}{R\,\delta l^2} \sum_\lambda \left[(E_{\lambda-1}^{-1} E_\lambda - 1)\, N_{u+}^\lambda + (E_{\lambda+1}^{-1} E_\lambda - 1)\, N_{u+}^\lambda \right] P \\ & - \frac{1}{R\,\delta l^2} \sum_\lambda \left[(E_{\lambda-1} E_\lambda^{-1} - 1)\, N_{u-}^\lambda + (E_{\lambda+1} E_\lambda^{-1} - 1)\, N_{u-}^\lambda \right] P \\ & + \frac{1}{2} \frac{\delta u}{\delta l} \left\{ \sum_\lambda (E_{\lambda+1}^{-1} E_\lambda - 1)\, \tfrac{1}{2} \mid N_u^\lambda \mid (\mid N_u^\lambda \mid -1) P \right. \\ & \left. + \sum_\lambda (E_\lambda^{-1} E_{\lambda-1} - 1)\, \tfrac{1}{2} \mid N_u^\lambda \mid (\mid N_u^\lambda \mid -1) P \right\} \end{aligned} \tag{3}$$

where the shift operators E_λ are defined by

$$\begin{aligned} E_\lambda F(\ldots, N_u^\lambda, \ldots) &= F(\ldots, N_u^\lambda + 1, \ldots) \\ E_\lambda^{-1} F(\ldots, N_u^\lambda, \ldots) &= F(\ldots, N_u^\lambda - 1, \ldots) . \end{aligned} \tag{4}$$

By construction, the equations for the expectation values $\delta u \langle N_u^\lambda \rangle$ obtained from the above master equation by taking the 'thermodynamic limit'

$$\delta u \longrightarrow 0 \qquad \text{and} \qquad \delta u N_u^\lambda = \text{constant} \tag{5}$$

and the limit $\delta l \longrightarrow 0$ lead to Burgers' equation (1).

The master equation (3) was constructed within a many-particle picture. Since the stochastic process N_u^λ is integer valued it can be interpreted as the number of fictitious velocity particles situated in cell λ. Furthermore, the transitions of the states of the fluid may be viewed as one-particle jumps from cell λ to a neighbouring one.

Within this picture of a many-particle system, a positive number $N_u^\lambda > 0$ is regarded as the number N_{u+}^λ of velocity particles; likewise, a negative number $N_u^\lambda < 0$ is interpreted as the number $| N_{u-}^\lambda |$ of antiparticles of velocity, i.e. we have $N_u^\lambda = N_{u+}^\lambda + N_{u-}^\lambda$. The fundamental importance of this picture immediately appears by interpreting the different terms in the master equation. Namely, the first two terms on the right-hand side describe the diffusion of a collective system of velocity particles and antiparticles, that is, a collective random walk, whereas the last two terms represent the convection of velocity particles and antiparticles induce a nonlinear self-interaction of the latter. The rate at which one velocity particle jumps from one cell λ to a neighbouring one is proportional to the number of pairs of velocity particles N_u^λ in the cell λ. Summing up, the many-particle picture allows the formulation of diffusion, as well as convection, in terms of one-step processes where each particle or antiparticle may jump only to a neighbouring cell.

It should be noted that the master equation (3) is an equation for the transition probability of the stochastic process $N_u^\lambda(t)$. Only by fixing an appropriate initial probability distribution can one extract from it information about the probability distribution of different states of the system. Thus, this formalism naturally allows one to impose initial conditions in a probabilistic sense.

Let us now turn our attention to the application of the mesoscopic description of the dynamics of fluids to the problem of turbulence. In order to model homogeneous turbulence it is first necessary to implement appropriate boundary and initial conditions of the velocity. Firstly, we impose periodic boundary conditions by dividing the interval $[0,2]$ in a number $M+1$ of cells labelled by the index $\lambda = 0,1,2,\ldots,M$ (i.e. $\delta l = 2/(M+1)$) and by identifying the number N_u^{M+1} of velocity particles in the $(M+1)$th cell with the number of particles N_u^0 in the 0th cell, that is

$$N_u^{M+1}(t) \equiv N_u^0(t) . \tag{6}$$

The statistical initial-value problem typical for homogeneous turbulence states that the random variables $\delta u\, N_u^\lambda(0)$ should be uniformly distributed between the normalized velocity maxima -1 and $+1$. By introducing an initial correlation length l an appropriate initial condition reads

$$N_u^{\sigma+k}(0) = \mathrm{int}\left(\frac{\eta_\sigma}{\delta u}\right) \qquad k = 0,1,2,\ldots,l-1 \tag{7}$$

where the η_σ for $\sigma = 0,l,2l,\ldots$, are independent random numbers uniformly distributed over the interval $[-1,+1]$ and the function $\mathrm{int}(y)$ denotes the integer part of y.

It is clear, that as a consequence of the imposed initial condition (7) the mean value of the velocity (2) is identically zero. Thus, the physical properties expressing the character of homogeneous turbulence are directly related to higher moments of the stochastic process $N_u^\lambda(t)$, and specially to the fluctuations naturally arising in the system. In our numerical investigations we studied the time development of the velocity correlation function $Q(\mu,t)$ which in our formalism is consequently defined as a second moment of the stochastic process $N_u^\lambda(t)$

$$Q(\mu,t) := \delta u^2 \langle N_u^{\lambda+\mu}(t) N_u^\lambda(t) \rangle . \tag{8}$$

The correlation $Q(\mu, t)$ is computed assuming spatial homogeneity by taking the space average

$$Q(\mu, t) = \frac{\delta u^2}{M+1} \sum_{\lambda=0}^{M} \langle N_u^{\lambda+\mu}(t) N_u^{\lambda}(t) \rangle . \tag{9}$$

It might be interesting to note that the so-called Loitsiansky constant [1] C_L which is defined in our formalism by the equation

$$C_L := \frac{1}{M+1} \sum_{\mu=0}^{M} Q(\mu, t) \tag{10}$$

is exactly conserved. This follows from the fact that C_L is proportional to the expectation value of the square of the total number of velocity particles. This total particle number remains strictly constant since the above master equation does not contain any source or sink for velocity particles. Furthermore, it can be shown from the master equation (3) that Taylor's relation [1] which in our formalism reads

$$\frac{\partial}{\partial t} \sum_{\lambda=0}^{M} \delta l \, \tfrac{1}{2} \delta u^2 \langle (N_u^{\lambda})^2 \rangle = \frac{1}{R} \sum_{\lambda=0}^{M} \delta l \, \frac{\delta u^2}{\delta l^2} \langle N_u^{\lambda} (N_u^{\lambda+1} - 2N_u^{\lambda} + N_u^{\lambda-1}) \rangle \tag{11}$$

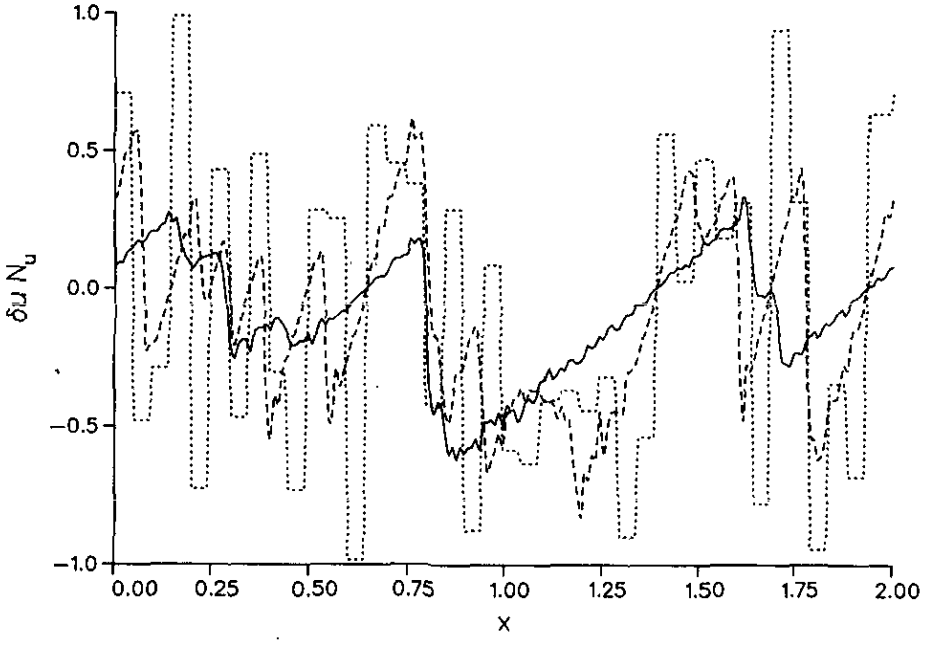

Figure 1. Time development of a realization of the stochastic process N_u^{λ} underlying the master equation (3) with random initial condition (7) for the Reynolds number $R = 1000$. The realization is shown at three different times: $t_0 = 0.0$ (dotted curve), $t_1 = 0.2$ (broken curve), and $t_4 = 0.8$ (full curve). For the simulation we have chosen $\delta u = 0.001$, $M = 200$, and $l = 5$. Note, that in this picture $x = \delta l \, \lambda = 2\lambda/(M+1)$, where $\lambda = 0, 1, 2, \ldots, M+1$.

is also satisfied up to terms of order $O(\delta l^2)$. Finally, an important quantity is the energy spectrum given by the equation

$$E(k,t) = \frac{1}{M+1} \sum_{\mu=0}^{M/2} Q(\mu,t) \cos(\pi k \mu) . \tag{12}$$

Having fixed the boundary and the initial conditions the dynamics of the system can now be studied by means of the method of stochastic simulation using stochastic time steps [7, 9]. This method is particularly suited for the investigation of high-dimensional systems governed by master equations. In stochastic simulations one generates a sufficiently large number of realizations of the stochastic process underlying the master equation (3). The physical quantities defined above are then evaluated as time dependent ensemble averages.

In figure 1 we display one realization of the stochastic process defined by the master equation (3) and the random initial condition (7) for the Reynolds number $R = 1000$ for three different times: $t_0 = 0.0$, $t_1 = 0.2$, and $t_4 = 0.8$. The initial random velocity configuration $\{N_u^{\lambda}(0)\}$ develops a typical shock wave structure consisting of smoothly increasing ramps followed by sharp shocks. For longer times, the velocity decays due to dissipation.

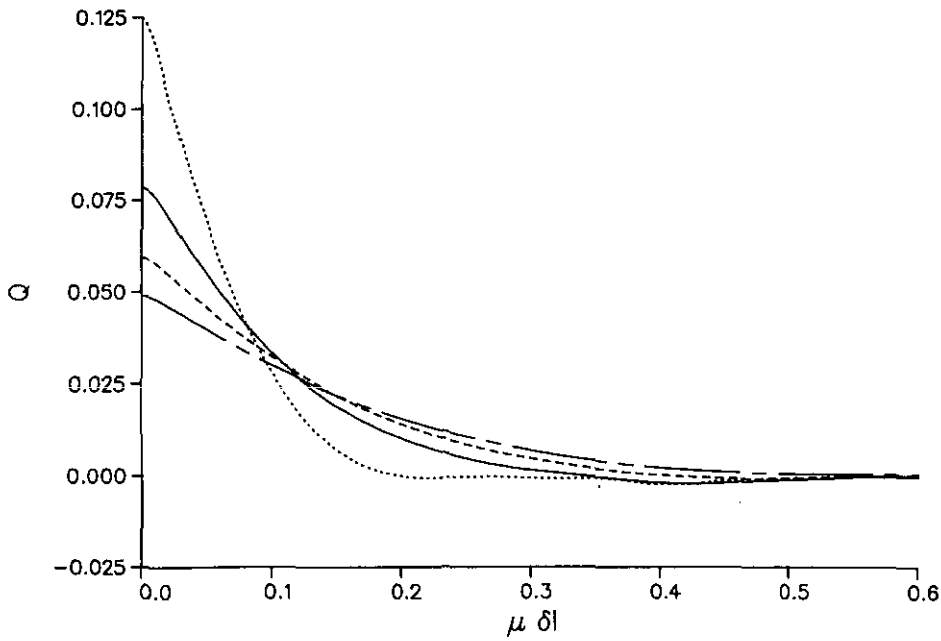

Figure 2. The velocity correlation function $Q(\mu, t)$ for the Reynolds number $R = 1000$ evaluated according to equation (9) by averaging over 150 realizations for four different times: $t_1 = 0.2$ (dotted curve), $t_2 = 0.4$ (full curve), $t_3 = 0.6$ (broken curve), and $t_4 = 0.8$ (chain curve). For the stochastic simulation we have chosen $M = 200$, $l = 5$, and $\delta u = 0.001$.

In figure 2 we show the correlation function evaluated according to (9) for four different times: $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$, and $t_4 = 0.8$. As can be seen, the correlation function exhibits the typical properties which are known from the theory

of homogeneous turbulence. Furthermore, we numerically checked Taylor's relation by determining the energy dissipated between the times $t = 0.7$ and $t = 0.8$ and by comparing it with the corresponding time integral of the curvature of the correlation function. The mean relative deviation between these two quantities was found to be less than 1%.

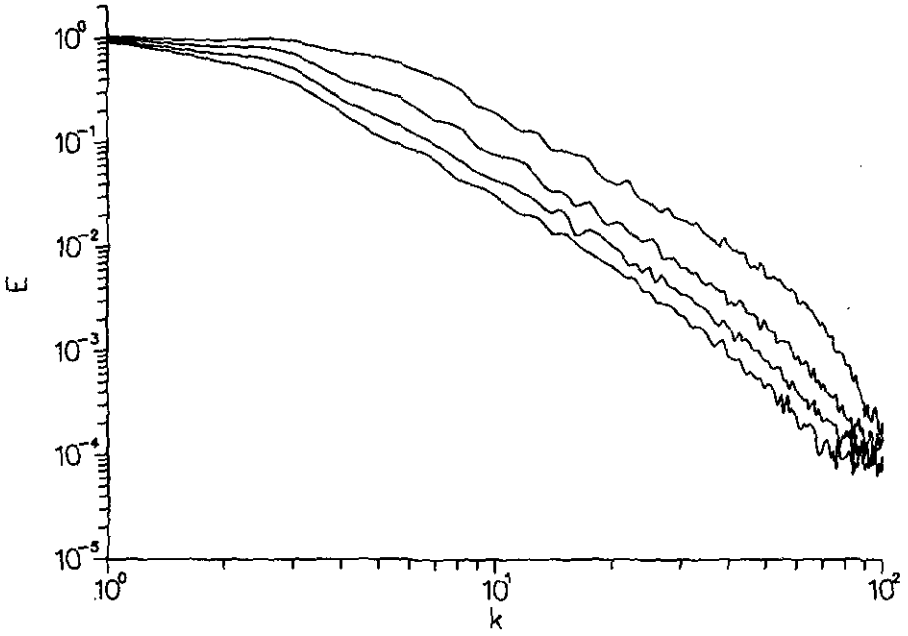

Figure 3. The normalized energy spectrum $E(k,t)/E(0,t)$ computed according to equation (12) for four different times: $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$, and $t_4 = 0.8$. The parameters of the simulation were chosen as in the preceding figure.

In figure 3 we present the normalized energy spectrum $E(k,t)/E(0,t)$ for $R = 1000$ at the above times t_1, t_2, t_3, and t_4 as a function of the wavenumber k. The typical power law behaviour of the energy spectrum, $E(k) \sim k^{-2}$, can be observed. This well known theoretical fact follows directly from the shock structure and reflects the presence of an energy cascade in this model.

In this letter we have shown how to attack the problem of homogeneous turbulence within the framework of a recently proposed mesoscopic description of fluid dynamics. Thus, a discrete stochastic process has been constructed which allows a purely probabilistic description of Burgers' turbulence model. Having properly defined physical quantities as moments of the underlying stochastic process the mesoscopic approach reproduces well known theoretical results. Obviously, the same approach can be applied to turbulent fluid motion in higher dimensions as will be shown in later work.

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